关于数分中的大欧与小欧：

(一) 定义：设 $x \to x_0$ 时，$\alpha(x)$，$\beta(x)$ 都是无穷小量，或都是无穷大量.

(1). 若 $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)} = 0$，则称 $\alpha(x) = \overset{\text{小欧}}{o}(\beta(x))$ $(x \to x_0)$，此时，有

$\dfrac{\alpha(x)}{\beta(x)} = \dfrac{o(\beta(x))}{\beta(x)} \longrightarrow 0$ $(x \to x_0)$. 即小欧表示高阶无穷小量.

(2). 若在 x_0 附近 $\dfrac{\alpha(x)}{\beta(x)}$ 有界：$\exists M > 0$，使 $\left|\dfrac{\alpha(x)}{\beta(x)}\right| \le M$. 则称

$\alpha(x) = \overset{\text{大欧}}{O}(\beta(x))$，此时，有

$\left|\dfrac{\alpha(x)}{\beta(x)}\right| = \left|\dfrac{O(\beta(x))}{\beta(x)}\right| \le M$. 即 $\dfrac{O(\beta(x))}{\beta(x)}$ 是有界量，而 $\dfrac{o(\beta(x))}{\beta(x)}$ 是无穷小.

(二). 小欧 o 与大欧 O 的关系：

(1). 小欧 o 是大欧 O 的一个真子集. 理由：

当 $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)} =$ 常数 0 时，根据有极限必有界可知，$\exists M > 0$，

使 $\left|\dfrac{\alpha(x)}{\beta(x)}\right| \le M$. 由此，可知当 $\alpha(x) = o(\beta(x))$ 时必有 $\alpha(x) = O(\beta(x))$

再由有界的未必有极限可知，小欧 o 是大欧 O 的一个真子集。

(2). 当 $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)} = C \in R$，$C \ne 0$ 时，称 $\alpha(x)$ 与 $\beta(x)$ 是同级(阶)无穷小.

(大). 此时，必有 $\left|\dfrac{\alpha(x)}{\beta(x)}\right| \le M$，$M > 0$. $\therefore \alpha(x) = O(\beta(x))$，$x \to x_0$.

综上可知：在 $x\to x_0$ 时，若 $\alpha(x)$ 是 $\beta(x)$ 的高阶无穷小，或同级无穷小，或等价无穷小，都可统一记作：$\alpha(x)=O(\beta(x))$ 一大欧.

(3). 有限个小欧的代数和仍是小欧.

证：设 $o_1(\alpha(x)), o_2(\alpha(x)), \cdots, o_m(\alpha(x))$ 都是 $x\to x_0$ 时的 $\alpha(x)$ 的高阶无穷小量，即 $\frac{o_1(\alpha(x))}{\alpha(x)}\to 0, \cdots, \frac{o_m(\alpha(x))}{\alpha(x)}\to 0$ $(x\to x_0)$. 则因

$$\frac{o_1(\alpha(x))+o_2(\alpha(x))+\cdots+o_m(\alpha(x))}{\alpha(x)}\longrightarrow 0+0+\cdots+0=0 \quad (x\to x_0).$$

$\therefore o_1(\alpha(x))+o_2(\alpha(x))+\cdots+o_m(\alpha(x))=o(\alpha(x))$.

(4). 有限个大欧的代数和仍是大欧.

证：设 $\frac{O_1(\alpha(x))}{\alpha(x)}, \frac{O_2(\alpha(x))}{\alpha(x)}, \cdots, \frac{O_m(\alpha(x))}{\alpha(x)}$ 都是 $x\to x_0$ 时的有界量.

则从 $\left|\frac{O_1(\alpha(x))+O_2(\alpha(x))+\cdots+O_m(\alpha(x))}{\alpha(x)}\right|\le\left|\frac{O_1(\alpha(x))}{\alpha(x)}\right|+\left|\frac{O_2(\alpha(x))}{\alpha(x)}\right|+\cdots+\left|\frac{O_m(\alpha(x))}{\alpha(x)}\right|$

$\le M_1+M_2+\cdots+M_m\triangleq M$. $(M_1>0,\cdots,M_m>0)$ 可知.

$O_1(\alpha(x))+O_2(\alpha(x))+\cdots+O_m(\alpha(x))=O(\alpha(x))$

(5). 小欧与大欧的和差仍是大欧.

证：设 $o(\alpha(x))\pm O(\alpha(x))$ 是小欧与大欧的和差. 则从

$$\left|\frac{o(\alpha(x))\pm O(\alpha(x))}{\alpha(x)}\right|\le\left|\frac{o(\alpha(x))}{\alpha(x)}\right|+\left|\frac{O(\alpha(x))}{\alpha(x)}\right|\le M_1+M_2 \quad (M_1>0, M_2>0)$$

可知，$o(\alpha(x))\pm O(\alpha(x))=O(\alpha(x))$

(6). 小欧与大欧之积必是小欧：

证：从 $\frac{o(\alpha(x))\,O(\alpha(x))}{\alpha(x)} = \frac{o(\alpha(x))}{\alpha(x)}\cdot\frac{O(\alpha(x))}{\alpha(x)}\cdot\alpha(x)\longrightarrow 0\ (x\to x_0)$

（无穷小量与有界变量之积是无穷小量）可知：

$o(\alpha(x))\,O(\alpha(x)) = o(\alpha(x))$

(7). 小欧与大欧的复合必是小欧：

证：(1) 从 $\frac{o(O(\alpha(x)))}{\alpha(x)} = \frac{o(O(\alpha(x)))}{O(\alpha(x))}\cdot\frac{O(\alpha(x))}{\alpha(x)}$ 是无穷小与有界量之积 $\longrightarrow 0\ (x\to x_0)$.

$\Rightarrow o(O(\alpha(x))) = o(\alpha(x))\quad (x\to x_0)$.

(2) 从 $\frac{O(o(\alpha(x)))}{\alpha(x)} = \frac{O(o(\alpha(x)))}{o(\alpha(x))}\cdot\frac{o(\alpha(x))}{\alpha(x)}$ 是有界量与无穷小之积 $\longrightarrow 0\ (x\to x_0)$

$\Rightarrow O(o(\alpha(x))) = o(\alpha(x))\quad (x\to x_0)$.

(8). 从 $\lim\limits_{x\to+\infty}\frac{a^x}{x^x} = 0 = \lim\limits_{x\to+\infty}\frac{x^m}{a^x} = \lim\limits_{x\to+\infty}\frac{\ln x}{x^m}\quad (\forall a>1, m\in N^*)$

知：$a^x = o(x^x)$；$x^m = o(a^x)$；$\ln x = o(x^m)$. $x\to+\infty, \forall a>1, \forall m\in N^*$

(9). 从 $\lim\limits_{n\to\infty}\frac{n!}{n^n} = 0 = \lim\limits_{n\to\infty}\frac{a^n}{n!} = \lim\limits_{n\to\infty}\frac{n^m}{a^n} = \lim\limits_{n\to\infty}\frac{\ln n}{n^m}\quad (\forall a>1, m\in N^*)$

知：$n! = o(n^n)$；$a^n = o(n!)$；$n^m = o(a^n)$，$\ln n = o(n^m)$，$(\forall a>1, m\in N^*)$