# 习题课讲义

Def：称 $f$ 在 $[a,b]$ 上可积，记 $f\in R[a,b]$，指 $\|T\|\to 0$ 时 $S_n(T)$ 极限存在且与 $x_i$, $\xi_i$ 选取无关。极限定义为 $f(x)$ 在 $[a,b]$ 上积分，记为 $\int_a^b f(x)\,dx$

Rmk：$f(x)$ 连续 / 有限个第一类间断点 / 有限个有界震荡间断点 $\Rightarrow$ 可积

$f(x)$ 在 $[a,b]$ 无界 $\Rightarrow$ 不可积

例：有原函数 + 不连续 + 可积

$$F(x)=\begin{cases} x^2\sin\frac{1}{x} & 0<x\le 1 \\ 0 & x=0 \end{cases} \qquad f(x)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x} & 0<x\le 1 \\ 0 & x=0 \end{cases}$$

有原函数 + 不连续 + 不可积

$$F(x)=\begin{cases} x^2\sin\frac{1}{x^2} & 0<x\le 1 \\ 0 & x=0 \end{cases} \qquad f(x)=\begin{cases} 2x\sin\frac{1}{x^2}-\frac{2}{x}\cos\frac{1}{x^2} & 0<x\le 1 \\ 0 & x=0 \end{cases}$$

例：极限与积分不能交换的例子

$f(x)=0$，$f_n(x)=2^n\chi_{\{1-\frac{1}{2^{n-1}}<x<1-\frac{1}{2^n}\}}$

则 $\int_0^1 f(x)\,dx=0$，$\int_0^1 f_n(x)\,dx=1$，$\lim f_n(x)=f(x)$。

但 $\lim\limits_{n\to\infty}\int_0^1 f_n(x)\,dx \ne \int_0^1 \lim f_n(x)\,dx$

含参积分求导：$F(y)=\int_{a(y)}^{b(y)} f(x,y)\,dx$

$$F'(y)=\int_{a(y)}^{b(y)} f_y(x,y)\,dx+f(b(y),y)\,b'(y)-f(a(y),y)\,a'(y)$$

补充例题：

1. 设 $f(x)\in C^2[0,2]$，$f'(0)=f'(2)=0$，Pf：$\exists\,\xi\in(0,2)$，使得 $\int_0^2 f(x)dx=f(0)+f(2)+\frac{1}{3}f''(\xi)$

Proof：令 $g(x)=\int_0^x f(t)dt$. 则

$$g(1)=g(0)+g'(0)+\frac{g''(0)}{2}+\frac{g'''(\xi)}{6}.$$

$$g(1)=g(2)+g'(2)+\frac{g''(2)}{2}+\frac{g'''(\eta)}{6}$$

则 $f(0)+\frac{1}{6}f''(\xi)=\int_0^2 f(x)dx-f(2)-\frac{1}{6}f''(\eta)$

由二阶导连续. 有

$$\int_0^2 f(x)dx=f(0)+f(2)+\frac{1}{6}\left(f''(\xi)+f''(\eta)\right)=f(0)+f(2)+\frac{1}{3}f''(\xi)$$

2. Lemma：$\int_0^a\left(\int_0^x f(t)dt\right)dx=\int_0^a f(x)(a-x)dx$

$$\int_0^a\left(\int_0^x f(t)dt\right)dx = x\int_0^x f(t)dt\Big|_0^a-\int_0^a x\,d\left(\int_0^x f(t)dt\right)$$

$$=a\int_0^a f(t)dt-\int_0^a xf(x)dx$$

$$=\int_0^a f(x)(a-x)dx$$

（交换积分次序）$\int_0^\pi\left(\int_0^x\frac{\sin t}{\pi-t}dt\right)dx=\int_0^\pi\frac{\sin x}{\pi-x}(\pi-x)dx=\int_0^\pi\sin x\,dx=2$

3.（对称性）$\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x}{1+\sqrt{\tan x}}dx$

令 $x=\frac{\pi}{2}-t$. 则有 $\int_0^{\frac{\pi}{2}}\frac{\sin x\cdot\cos x}{1+\sqrt{\tan x}}dx=\int_{\frac{\pi}{2}}^0\frac{\sin(\frac{\pi}{2}-t)\cdot\cos(\frac{\pi}{2}-t)}{1+\sqrt{\tan(\frac{\pi}{2}-t)}}d(\frac{\pi}{2}-t)$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin t\cos t\,\sqrt{\tan t}}{1+\sqrt{\tan t}}dt$$

故 $\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x}{1+\sqrt{\tan x}}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin x\cos x\,dx=\frac{1}{4}$

（对称性与单调性）：

$f(x)\in[a,b]$ 连续单增. 证明：$\int_a^b xf(x)dx\geqslant\frac{a+b}{2}\int_a^b f(x)dx$

Proof：$\Leftrightarrow\int_a^b\left(x-\frac{a+b}{2}\right)f(x)dx$. $\qquad x-\frac{a+b}{2}=t$

$\Leftrightarrow\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}}t\,f\left(t+\frac{a+b}{2}\right)dt\geqslant0$

令 $h=\frac{b-a}{2}$. $k=\frac{a+b}{2}$. 则

$$LHS = \int_{-h}^{h} t f(t+k)\,dt = \int_0^h t f(t+k)\,dt + \int_{-h}^{0} t f(t+k)\,dt$$

$$= \int_0^h t\,[f(k+t)-f(k-t)]\,dt \geq 0$$

4. $f(x)$ 在 $[0,1]$ 上非负连续. $f^2(x) \leq 1+2\int_0^x f(t)\,dt$. 求证: $f(x) \leq 1+x$.

Proof: 令 $g(x)=f(x)-x$. 则

$$g^2(x)+2xg(x)+x^2 \leq 1+2\int_0^x g(t)\,dt + x^2$$

$$g^2(x)-1 \leq 2\left(\int_0^x g(t)\,dt - xg(x)\right)$$

$f$ 连续 $\Rightarrow$ $g$ 连续. 令 $x_0$ 为 $g(x)$ 在 $[0,1]$ 上最大值点. 则

$$g^2(x_0)-1 \leq 2\left(\int_0^{x_0} g(t)\,dt - x_0 g(x_0)\right) \leq 2x_0\,(g(\xi)-g(x_0)) \leq 0$$

$$\Rightarrow \quad g(x) \leq g(x_0) \leq 1 \qquad (g(x) \geq -1)$$

5. $f(x)$ 在 $R$ 上连续. $f(x)\int_0^x f(t)\,dt$ 在 $R$ 上单调递减, 求证: $f(x)$ 为常值.

Proof: ① $f(x)=c$. 则 $f(x)\int_0^x f(t)\,dt = c^2x$ 单减 $\Rightarrow$ $c=0$.

即 $f(x)\equiv 0$

② 若 $f(0)>0$. 令 $\varepsilon=\frac{f(0)}{2}>0$. $\exists\delta$. $\forall x\in(-\delta,\delta)$. $f(x)>f(0)-\varepsilon=\frac{f(0)}{2}>0$.

$\forall x\in(0,\delta)$. $f(x)\int_0^x f(t)\,dt > \frac{f(0)}{2}\cdot\frac{f(0)}{2}x = \frac{f^2(0)}{4}x > 0 = f(0)\int_0^0 f(t)\,dt$

与单减矛盾. $f(0)<0$ 亦然. $\Rightarrow f(0)=0$

③ $\varphi(x)=f(x)\int_0^x f(t)dt=\left[\left(\frac{1}{2}\int_0^x f(t)dt\right)^2\right]'=F'(x)$.

则 $F'(x)\downarrow$. $F'(0)=0$. 故 0 为 $F(x)$ 极大值点

$0\le F(x)\le F(0)=0 \Rightarrow F(x)\equiv 0 \Rightarrow \int_0^x f(t)dt\equiv 0$.

故 $f(x)=\left(\int_0^x f(t)dt\right)'=0$

6. $f(x)\in C'[0,1]$, $f(0)=0$. 求证: $\int_0^1 |f(x)|^2dx\le\frac{1}{2}\int_0^1(1-x^2)|f'(x)|^2dx$. 仅在 $f(x)\equiv cx$ 处取等

Proof. $f^2(x)=\left(\int_0^x f'(t)dt\right)^2=\left(\int_0^x f'(t)\cdot 1\,dt\right)^2\le\int_0^x(f'(t))^2dt\cdot\left(\int_0^x 1\,dt\right)=x\int_0^x(f'(t))^2dt$

两边积分. $\int_0^1 f^2(x)dx\le\int_0^1 x\int_0^x f'^2(t)dt\,dx=\int_0^1\left(\int_0^x f'(t)^2dt\right)d\frac{x^2}{2}$

$=\frac{1}{2}\int_0^1 f'(t)^2dt-\frac{1}{2}\int_0^1 f'(t)^2t^2dt=\frac{1}{2}\int_0^1(1-x^2)f'(x)^2dx$

取等为 $f'=\lambda 1 \Rightarrow f=cx$

7. $f(x)$ 在 $[0,1]$ 上有 $f(0)=f(1)=0$ 且连续可微 求证: $\left(\int_0^1 f(x)dx\right)^2\le\frac{1}{12}\int_0^1|f'(x)|^2dx$. 仅在 $f(x)=Ax(1-x)$ 处.

Hint: $\boxed{\text{考虑 Cauchy}}$.

由不等号方向: $\int_0^1 f'^2(x)dx\cdot\int_0^1 g^2(x)dx\ge\left(\int_0^1|f'(x)g(x)|dx\right)^2$.

再由取等: $f'(x)=cg(x) \longleftrightarrow$ 对比 $f(x)=Ax(1-x)$.

取 $A=1$. $g(x)=1-2x$. 即

Proof. $\int_0^1|f'(x)|^2dx\cdot\int_0^1(1-2x)^2dx\ge\left(\int_0^1 f'(x)(1-2x)dx\right)^2=\left(\int_0^1 1-2x\,d f(x)\right)^2$

$=\left(f(x)(1-2x)\Big|_0^1-\int_0^1 f(x)d(1-2x)\right)^2=4\left(\int_0^1 f(x)dx\right)^2$.

而 $\int_0^1(1-2x)^2dx=\frac{1}{3}$. 代入. 则

$\left(\int_0^1 f(x)dx\right)^2\le\frac{1}{12}\int_0^1|f'(x)|^2dx$

8 $f(x)\in C'[0,1]$, $f(0)=0$, $f(1)=1$. 求证: $\int_0^1|f(x)+f'(x)|dx\ge 1$

Proof: $\int_0^1|f(x)+f'(x)|dx=\int_0^1|(e^xf(x))'|e^{-x}dx\ge\frac{1}{e}\int_0^1(e^xf(x))'dx=\frac{1}{e}\,e^xf(x)\Big|_0^1=1$

9. $f(x)$ 在 $[0,1]$ 上连续，且 $0\le f(x)\le 1$. 求证：$2\int_0^1 x f(x)dx \ge (\int_0^1 f(x)dx)^2$，并求取等.

Proof: $f(x)\in C[0,1] \Rightarrow F(x)=\int_0^x f(t)dt$ 可导.

令 $G(x) = 2\int_0^x t f(t)dt - (\int_0^x f(t)dt)^2$

$G'(x) = 2x f(x) - 2f(x)\int_0^x f(t)dt = 2f(x)(x-\int_0^x f(t)dt) \ge 0$

故 $G(1)\ge G(0)=0$. 成立.

取等：$G(1)=G(0)=0 \Rightarrow G'(x)=0$. 即 $\int_0^x f(t)dt = x$ 或 $f(x)=0 \Rightarrow f(x)\equiv 0$ 或 $f(x)\equiv 1$

10. $f(x)$ 在 $[0,1]$ 上可微. $x\in(0,1)$ 时 $f'(x)\in(0,1)$，$f(0)=0$，试证：$(\int_0^1 f(x)dx)^2 > \int_0^1 f^3(x)dx$

Proof: 令 $G(x) = (\int_0^x f(t)dt)^2 - \int_0^x f^3(t)dt$

$G'(x) = f(x)\cdot[2\int_0^x f(t)dt - f^2(x)]$

令 $H(x) = 2\int_0^x f(t)dt - f^2(x)$

$H'(x) = 2f(x) - 2f(x)f'(x) = 2f(x)(1-f'(x)) > 0$

$\therefore H(x)\uparrow$. $H(x)\ge H(0)=0$

$\therefore G'(x)>0$. $G(1) > G(0)=0$

另解：$\dfrac{(\int_0^1 f(x)dx)^2}{\int_0^1 f^3(x)dx} > 1$

令 $F(x) = (\int_0^x f(t)dt)^2$，$G(x)=\int_0^x f^3(t)dt$.

$\dfrac{F(1)}{G(1)} = \dfrac{F(1)-F(0)}{G(1)-G(0)} = \dfrac{F'(\xi)}{G'(\xi)} = \dfrac{2\int_0^\xi f(t)dt}{f^2(\xi)}$ 再来一次！ $\xi\in(0,1)$

$= \dfrac{2f(\eta)}{2f(\eta)f'(\eta)} = \dfrac{1}{f'(\eta)} > 1$ $\eta\in(0,\xi)$

11. $f(x), g(x)\in C[a,b]$，单增. 证明：$\int_a^b f(x)dx\cdot\int_a^b g(x)dx \le (b-a)\int_a^b f(x)g(x)dx$.

Proof: 令 $F(u) = (u-a)\int_a^u f(x)g(x)dx - \int_a^u f(x)dx\cdot\int_a^u g(x)dx$

$f,g\in C[a,b] \Rightarrow F(u)$ 可微

$F'(u) = \int_a^u f(x)g(x)dx + (u-a)f(u)g(u) - g(u)\int_a^u f(x)dx - f(u)\int_a^u g(x)dx$

$= \int_a^u (f(u)-f(x))(g(u)-g(x))dx \ge 0$

$\therefore F(b) \geq F(a) = 0$.

12 $\int_0^{+\infty} x^3 e^{-x^2} dx$

$$\int_0^{\infty} x^3 e^{-x^2} dx = -\frac{1}{2}\int_0^{+\infty} x^2 de^{-x^2} = -\frac{1}{2}x^2 e^{-x^2}\Big|_0^{+\infty} + \frac{1}{2}\int_0^{+\infty} e^{-x^2}\, 2x\, dx$$

$$= -\frac{1}{2}\int_0^{+\infty} de^{-x^2} = -\frac{1}{2}e^{-x^2}\Big|_0^{+\infty} = \frac{1}{2}$$

13. $\lim\limits_{n\to\infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}}$ . $p > 0$.

$$\lim_{n\to\infty} \frac{(\frac{1}{n})^p + (\frac{2}{n})^p + \dots + (\frac{n}{n})^p}{n} = \int_0^1 x^p dx = \frac{1}{p+1}$$

14. pf: $\int_0^{\sqrt{2\pi}} \sin x^2 dx > 0$

Proof: 令 $x^2 = y$.

$$I = \int_0^{2\pi} \sin y \cdot \frac{1}{2\sqrt{y}} dy = \frac{1}{2}\int_0^{\pi} \frac{\sin y}{\sqrt{y}} dy + \frac{1}{2}\int_{\pi}^{2\pi} \frac{\sin y}{\sqrt{y}} dy = I_1 + I_2$$

$I_2$ 中 $z = y - \pi$. $I_2 = \frac{1}{2}\int_{\pi}^{2\pi} \frac{\sin y}{\sqrt{y}} dy = -\frac{1}{2}\int_0^{\pi} \frac{\sin z}{\sqrt{z+\pi}} dz$

$$\therefore I = \frac{1}{2}\int_0^{\pi} \sin y \left( \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+\pi}} \right) dy > 0$$