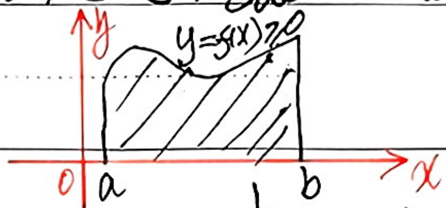
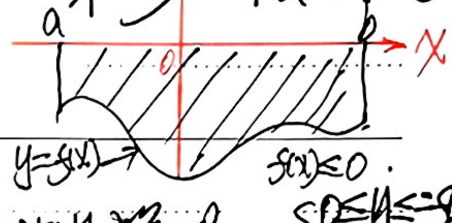
第30讲：定积分几何应用举例

（一）$\int_a^b f(x)dx$ 的几何意义：（设 $f\in R[a,b]$）

(1) 若 $f(x)\ge 0$，$x\in[a,b]$ 时，$\int_a^b f(x)dx$ 为曲边梯形 $D:\begin{cases}0\le y\le f(x)\\ a\le x\le b\end{cases}$ 的面积 $S(D)$；

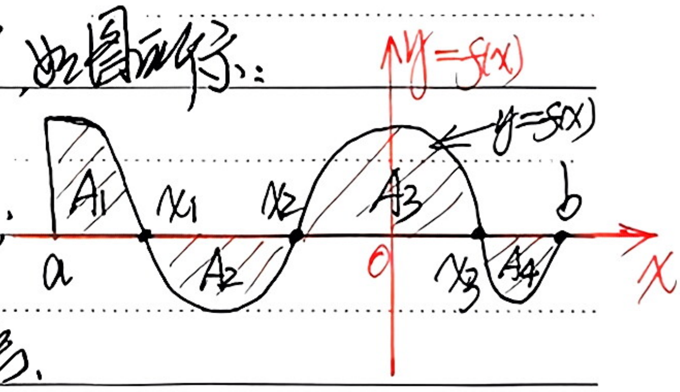




(2) 若 $f(x)\le 0$，$x\in[a,b]$ 时，$\int_a^b(-f(x))dx$ 是曲边梯形 $D:\begin{cases}0\le y\le -f(x)\\ a\le x\le b\end{cases}$ 的面积 $S(D)$，而 $\int_a^b f(x)dx=-\int_a^b(-f(x))dx=-S(D)$.

(3) 若 $f(x)$ 在 $[a,b]$ 上时正时负时，如图所示：

设 A_1,A_3 为 x 轴上方的两块面积，



A_2,A_4 为 x 轴下方的两块面积。

则 $A_1=\int_a^{x_1}f(x)dx$，$A_2=\int_{x_1}^{x_2}(-f(x))dx$，$A_3=\int_{x_2}^{x_3}f(x)dx$，$A_4=\int_{x_3}^{b}(-f(x))dx \Longrightarrow$

$$\int_a^b f(x)dx=\int_a^{x_1}f(x)dx+\int_{x_1}^{x_2}f(x)dx+\int_{x_2}^{x_3}f(x)dx+\int_{x_3}^{b}f(x)dx$$

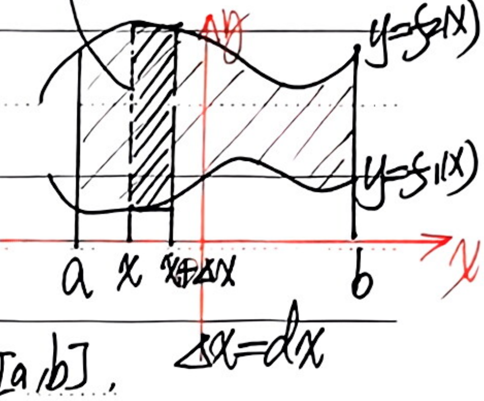
$$=\int_a^{x_1}f(x)dx-\int_{x_1}^{x_2}(-f(x))dx+\int_{x_2}^{x_3}f(x)dx-\int_{x_3}^{b}(-f(x))dx$$

$$=A_1-A_2+A_3-A_4.$$

即定积分的几何意义为"面积代数和"。x轴上方的为正，x轴下方的为负。

(三) 几何中平面区域D的面积A(D): $h(x)=f_2(x)-f_1(x)$

(1) 设D为 $\begin{cases} f_1(x) \le y \le f_2(x) \\ a \le x \le b \end{cases}$, 且 $f_1(x), f_2(x) \in C[a,b]$.



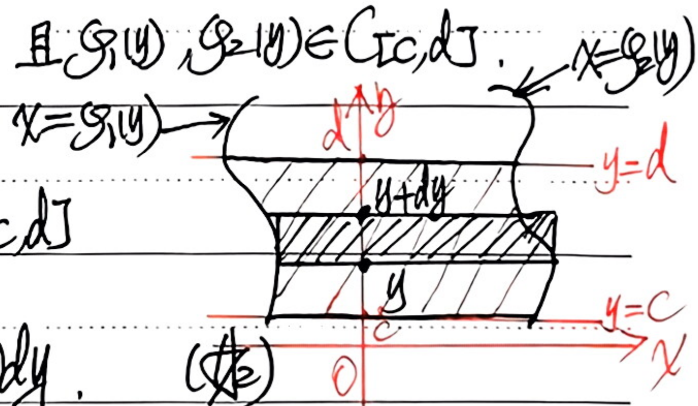
可采用"微元法"先算出 $[x, x+\Delta x] \subset [a,b]$

上的面积微元: $dA(x) = h(x)\cdot \Delta x$
$= (f_2(x)-f_1(x))\cdot dx$, $x\in[a,b]$.

再对面积微元 $dA(x)$ 作下定积分累加:

$$A(D) = \int_a^b dA(x) = \int_a^b (f_2(x)-f_1(x))\,dx \quad (f_2(x)\ge f_1(x)),$$ (式1)

(2) 设D为: $\begin{cases} \varphi_1(y) \le x \le \varphi_2(y) \\ c \le y \le d \end{cases}$, 且 $\varphi_1(y), \varphi_2(y) \in C[c,d]$.



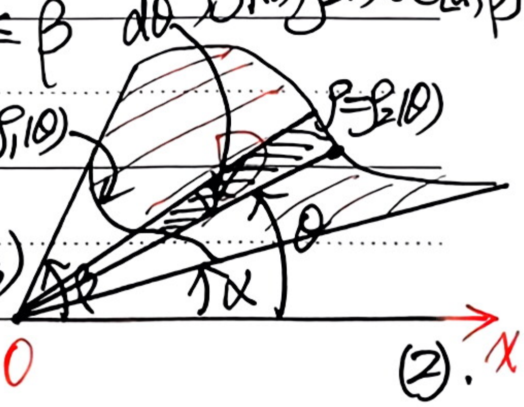
$\because dA(y) = (\varphi_2(y)-\varphi_1(y))\cdot dy$, $y\in[c,d]$

$$\therefore A(D) = \int_c^d dA(y) = \int_c^d (\varphi_2(y)-\varphi_1(y))\,dy.$$ (式2)

(3) 设D为极坐标形式: $\begin{cases} \rho_1(\theta) \le \rho(\theta) \le \rho_2(\theta) \\ \alpha \le \theta \le \beta \end{cases}$, $\rho_1(\theta), \rho_2(\theta) \in C[\alpha,\beta]$.

$\because dA(\theta) = \frac{1}{2}\rho_2^2(\theta)d\theta - \frac{1}{2}\rho_1^2(\theta)d\theta$, $\theta\in[\alpha,\beta]$,

$$\therefore A(D) = \int_\alpha^\beta dA(\theta) = \frac{1}{2}\int_\alpha^\beta (\rho_2^2(\theta)-\rho_1^2(\theta))\,d\theta,$$ (式3)



(2).

例1. 求下列曲线围成的平面区域D的面积$A(D)$:

(1). $D:\begin{cases} 0 \le y \le f(x) = \cos^n x \\ 0 \le x \le \frac{\pi}{2} \end{cases}$, $n \in N^*$

(注) 由华氏公式: $\int_0^{\frac{\pi}{2}} \cos^n x dx = \int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{(2m-1)!!}{(2m)!!} \times \frac{\pi}{2}, & n=2m,\ m \in N^* \\ \frac{(2m)!!}{(2m+1)!!} \times 1, & n=2m+1,\ m \in N \end{cases}$

(2). D是 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 所围的平面区域;

(3). D是心形线: $\rho(\theta)=a(1+\cos\theta)$ ($a>0$常数), $\theta\in[0,2\pi]$ 所围的平面区域.

解(1) $A(D)=\int_0^{\frac{\pi}{2}}(f(x)-0)dx=\int_0^{\frac{\pi}{2}}\cos^n x dx = \int_0^{\frac{\pi}{2}}\cos^{n-1}x d\sin x$

$=\sin x\cos^{n-1}x\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin x(n-1)\cos^{n-2}x(-\sin x)dx=0+\int_0^{\frac{\pi}{2}}(1-\cos^2 x)\cos^{n-2}x dx(n-1)$

$=\left(\int_0^{\frac{\pi}{2}}\cos^{n-2}x dx-\int_0^{\frac{\pi}{2}}\cos^n x dx\right)(n-1)=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}dx-(n-1)A(D)\Longrightarrow$

$A(D)=\int_0^{\frac{\pi}{2}}\cos^n x dx \triangleq I_n=\frac{n-1}{n}I_{n-2}=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot I_{n-4}=$

$\frac{n-1}{n}\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}I_{n-6}$, 当$n=2m$, $m\in N^*$时.

$I_{2m}=A(D)=\int_0^{\frac{\pi}{2}}\cos^{2m}x dx=\frac{2m-1}{2m}\frac{2m-3}{2m-2}\frac{2m-5}{2m-4}\cdots\times\frac{5}{6}\times\frac{3}{4}\times I_2$

而$I_2=\int_0^{\frac{\pi}{2}}\cos^2 x dx=\int_0^{\frac{\pi}{2}}\frac{1+\cos 2x}{2}dx=\left(\frac{1}{2}x+\frac{1}{4}\sin 2x\right)\Big|_0^{\frac{\pi}{2}}=\frac{1}{2}\times\frac{\pi}{2}$, (3).

$\because I_{2m}=\int_0^{\frac{\pi}{2}}\cos^{2m}x\,dx=\frac{(2m-1)(2m-3)(2m-5)\cdots5\times3\times1}{2m(2m-2)(2m-4)\cdots6\times4\times2}\times\frac{\pi}{2}$，$\forall m\in N^*$

$\triangleq\frac{(2m-1)!!}{(2m)!!}\times\frac{\pi}{2}$

当 $n=2m+1$ 时，有

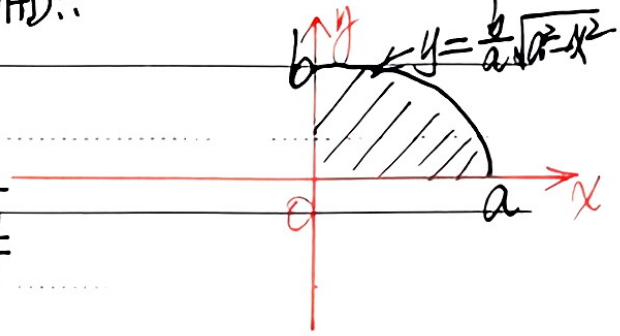
$I_{2m+1}=I_n=\frac{2m}{2m+1}\frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\cdots\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times I_1=\frac{(2m)!!}{(2m+1)!!}\times I_1$

而 $I_1=\int_0^{\frac{\pi}{2}}\cos x\,dx=\int_0^{\frac{\pi}{2}}\cos x\,dx=\sin x\Big|_0^{\frac{\pi}{2}}=1$.

另外，$\int_0^{\frac{\pi}{2}}\sin^n x\,dx\xlongequal[\text{则}dx=-dt]{\text{令}x=\frac{\pi}{2}-t}\int_{\frac{\pi}{2}}^0\sin^n(\frac{\pi}{2}-t)(-dt)=\int_0^{\frac{\pi}{2}}\cos^n t\,dt=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$

故有：$\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\begin{cases}\frac{(2m-1)!!}{(2m)!!}\times\frac{\pi}{2}，\text{当}n=2m, m\in N^*\\ \frac{(2m)!!}{(2m+1)!!}\times1，\text{当}n=2m+1, m\in N.\end{cases}$

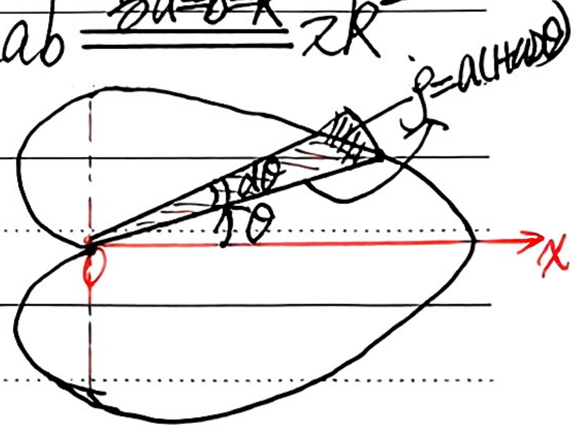
解(2). 椭圆域D的面积



$A(D)=4\int_0^a(\frac{b}{a}\sqrt{a^2-x^2}-0)dx\xlongequal[t\in[0,\frac{\pi}{2}]]{\text{令}x=a\sin t}$

$=4\int_0^{\frac{\pi}{2}}\frac{b}{a}\sqrt{a^2\cos^2t}\,d(a\sin t)=4\int_0^{\frac{\pi}{2}}\frac{b}{a}a^2\cos^2t\,dt$

$=4ab\int_0^{\frac{\pi}{2}}\cos^2t\,dt=4ab\times\frac{1!!}{2!!}\times\frac{\pi}{2}=\pi ab\xlongequal{\text{当}a=b=R}\pi R^2$.



解(3). $A(D)=\frac{1}{2}\int_0^{2\pi}\rho^2(\theta)d\theta$

$=\frac{1}{2}\int_0^{2\pi}a^2(1+\cos\theta)^2d\theta$

(4)

$=\frac{a^2}{2}\int_0^{2\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta=2a^2\int_0^{2\pi}\cos^4\frac{\theta}{2}d\theta \xrightarrow[\text{则 } d\theta=2du]{\text{令 } \frac{\theta}{2}=u} 2a^2\int_0^{\pi}(\cos^4 u)2du$

$=4a^2\int_0^{\pi}\cos^4 u du=4a^2\left(\int_0^{\frac{\pi}{2}}\cos^4 u du+\int_{\frac{\pi}{2}}^{\pi}\cos^4 u du\right)=4a^2\left(\int_0^{\frac{\pi}{2}}\cos^4 u du+\int_0^{\frac{\pi}{2}}\cos^4 u du\right)$

$=8a^2\int_0^{\frac{\pi}{2}}\cos^4 u du=8a^2\times\frac{3!!}{4!!}\times\frac{\pi}{2}=8a^2\times\frac{3\times1}{4\times2}\times\frac{\pi}{2}=\frac{3}{2}\pi a^2$.

注：通过换元积分法，可得：对 $\forall n\in N^*$，有

1°. $\int_0^{2\pi}\sin^{2n}x dx=\int_0^{2\pi}\cos^{2n}x dx=2\int_0^{\pi}\sin^{2n}x dx=4\int_0^{\frac{\pi}{2}}\sin^{2n}x dx=4\int_0^{\frac{\pi}{2}}\cos^{2n}x dx$;

2°. $\int_0^{2\pi}\sin^{2n+1}x dx=\int_0^{2\pi}\cos^{2n+1}x dx=0$, $n=0,1,2,3,\cdots$

(三) 平面光滑曲线 L 的弧长 S(L) 的计算方法：

(1) 设 $L: y=f(x)\in C^1[a,b]$，则弧长元素为

$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\sqrt{1+f'^2(x)}dx$, $x\in[a,b]$.

$S(L)=\int_a^b\sqrt{1+f'^2(x)}dx$ (公式)

(2) 设 $L:\begin{cases}x=x(t)\in C^1[\alpha,\beta]\\ y=y(t)\in C^1[\alpha,\beta]\end{cases}$，则 $\begin{cases}dx=x'(t)dt\\ dy=y'(t)dt\end{cases}$，$t\in[\alpha,\beta]$

$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{x'^2(t)+y'^2(t)}dt$, $t\in[\alpha,\beta]$.

$S(L)=\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)}dt$ (公式).

(3) 设L为极坐标方程：$\rho=\rho(\theta)$，$\rho(\theta)\in C^1[\alpha,\beta]$.

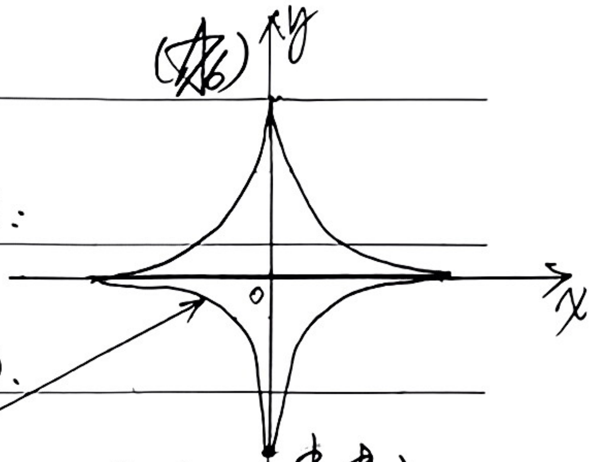
利用 $\begin{cases} x=\rho(\theta)\cos\theta \\ y=\rho(\theta)\sin\theta \end{cases} \Rightarrow x'^2_\theta+y'^2_\theta=(\rho'(\theta)\cos\theta-\rho(\theta)\sin\theta)^2+(\rho'(\theta)\sin\theta+\rho(\theta)\cos\theta)^2$

$=\rho^2(\theta)+\rho'^2(\theta) \Rightarrow$

故 $ds=\sqrt{x'^2(\theta)+y'^2(\theta)}\,d\theta=\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\,d\theta$，$\theta\in[\alpha,\beta] \Longrightarrow$

$S(L)=\int_\alpha^\beta\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\,d\theta$

(图6)



例2. 求下列曲线L的弧长$S(L)$：

(1). L为圆周：$x^2+y^2=R^2$ $(R>0)$.

(2). L为星形线：$\begin{cases} x=a\cos^3 t \\ y=a\sin^3 t \end{cases}$ $t\in[0,2\pi]$. $(a>0$, 常数$)$

(3). L为心脏线：$\rho(\theta)=a(1+\cos\theta)$，$\theta\in[0,2\pi]$ $(a>0$, 常数$)$

解(1)：设L为 $\begin{cases} x=R\cos t \\ y=R\sin t \end{cases}$，$t\in[0,2\pi]$. 则由对称性可知

$S(L)=4\int_0^{\frac{\pi}{2}}\sqrt{x'^2(t)+y'^2(t)}\,dt=4\int_0^{\frac{\pi}{2}}\sqrt{(-R\sin t)^2+(R\cos t)^2}\,dt$

$=4\int_0^{\frac{\pi}{2}}R\,dt=4R\times\frac{\pi}{2}=2\pi R$.

解(2) 由对称性可知：$S(L)=4\int_0^{\frac{\pi}{2}}\sqrt{x'^2(t)+y'^2(t)}\,dt$

$=4\int_0^{\frac{\pi}{2}}\sqrt{(3a\cos^2 t(-\sin t))^2+(3\sin^2 t\cos t\,a)^2}\,dt=4\times3a\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt$

$=12a\int_0^{\frac{\pi}{2}}\sin t\,d\sin t=6a\sin^2 t\Big|_0^{\frac{\pi}{2}}=6a$. (6).

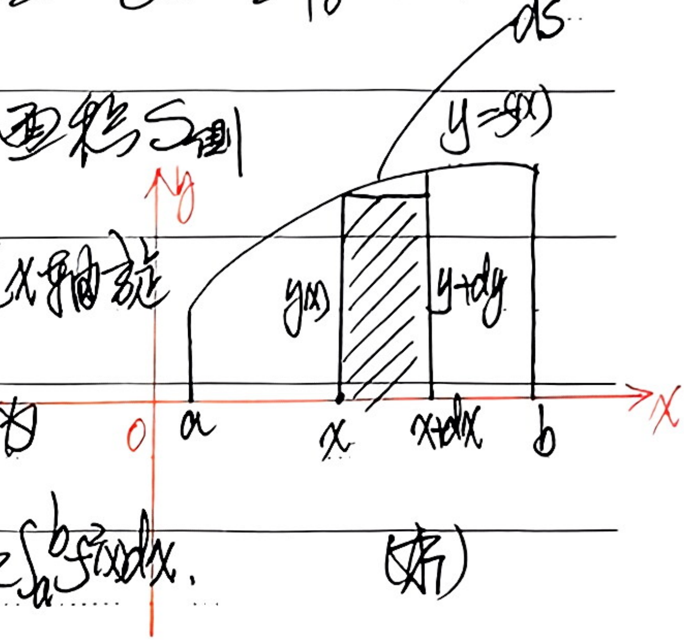
例(3). $\because \rho(\theta)=a(1+\cos\theta)$, $\rho'(\theta)=-a\sin\theta \Rightarrow \rho^2(\theta)+\rho'^2(\theta)=$

$a^2(1+2\cos\theta+\cos^2\theta+\sin^2\theta)=a^2(2+2\cos\theta)=2a^2(1+\cos\theta)=4a^2\cos^2\frac{\theta}{2}$.

由对称性可知，$s(L)=2\int_0^{\pi}\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\,d\theta=2\int_0^{\pi}\sqrt{4a^2\cos^2\frac{\theta}{2}}\,d\theta$

$=4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta=8a\int_0^{\pi}\cos\frac{\theta}{2}\,d\left(\frac{\theta}{2}\right)=8a\sin\frac{\theta}{2}\Big|_0^{\pi}=8a$.

(四) 旋转体Ω的体积与侧面积$S_{侧}$



曲边梯形$D:\begin{cases}0\le y\le f(x)\\ a\le x\le b\end{cases}$绕x轴旋转一周所形成的旋转体Ω的体积为

$V(\Omega)=\int_a^b \pi y^2(x)\,dx=\pi\int_a^b y^2(x)\,dx=\pi\int_a^b f^2(x)\,dx$. （公式）

在$[x,x+dx]$上，旋转体Ω的侧面积可近似看作圆台的侧面积＝(上底周长＋下底周长)×$\frac{1}{2}$×斜高$=\pi(y(x)+y(x)+dy)\,ds$

$=\pi(2y(x)+f'(x)dx)\sqrt{1+f'^2(x)}\,dx=2\pi f(x)\sqrt{1+f'^2(x)}\,dx+o(\Delta x)^2$

$\approx 2\pi f(x)\sqrt{1+f'^2(x)}\,dx=2\pi y\,ds$

故旋转体Ω的侧面积$S_{侧}=2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$ （公式）

例3. 求圆周$L:\begin{cases}x^2+y^2=R^2\ (R>0)\\ y\ge 0\end{cases}$绕x轴旋转一周所得(T).

求球体Ω的体积$V(\Omega)$与侧面积$S_侧$.

解：L的参数式：$\begin{cases} x=R\cos\theta \\ y=R\sin\theta, \end{cases}$ $\theta\in[0,2\pi]$.

(1°) $V(\Omega)=\pi\int_{-R}^{R}y^2(x)dx=\pi\int_{-R}^{R}(R^2-x^2)dx=2\pi\int_0^R(R^2-x^2)dx$

$=2(R^3-\frac{1}{3}R^3)\pi=\frac{4}{3}\pi R^3$.

(2°) $S_侧=2\pi\int_{-R}^{R}f(x)\sqrt{1+f'^2(x)}\,dx$　$f(x)=y=\sqrt{R^2-x^2}\Rightarrow$

$f(x)\sqrt{1+f'^2(x)}=\sqrt{R^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}=\sqrt{R^2-x^2}\frac{R}{\sqrt{R^2-x^2}}=R\Rightarrow$

$S_侧=2\pi\int_{-R}^{R}R\,dx=2\pi R(2R)=4\pi R^2$.

(2) 作业：ch5.3

1/(1),(2),(3)；2/(3)；3/(3)；4；5/(4)；ch5综/1.